Airy function - exact WKB results for potentials of odd degree

1999 J. Phys. A: Math. Gen. 32 1301

(http://iopscience.iop.org/0305-4470/32/7/020)

J. Phys. A: Math. Gen. **32** (1999) 1301–1311. Printed in the UK PII: S0305-4470(99)98783-8

Airy function—exact WKB results for potentials of odd degree

André Voros†
CEA–Saclay, Service de Physique Théorique, F–91191 Gif-Sur-Yvette Cedex, France

Received 29 October 1998

Abstract. An exact WKB (Wentzel–Kramers–Brillouin) treatment of 1D homogeneous Schrödinger operators (with the confining potentials q^N, N even) is extended to odd degrees N. The resulting formalism is first illustrated theoretically and numerically upon the spectrum of the cubic oscillator (potential $|q|^3$). Concerning the linear potential ($N = 1$), the theory exhibits a duality in which the Airy functions Ai, Ai′ become paired with the spectral determinants of the quartic oscillator ($N = 4$). Classic identities for the Airy function, as well as some less familiar ones, appear in this new perspective as special cases in a general setting.

A number of quantum spectral properties have now become established (by means of WKB theory, asymptotic or exact) for homogeneous q^N potentials on the real line having a discrete spectrum (confining case, i.e., N even). After recalling the required results (section 1), this work will mainly extend and apply the framework to odd powers N (section 2), then discuss, in particular, the resulting picture for the Airy function when $N = 1$ (section 3).

1. Summary of previous results

See [1] for a more detailed review.

Definitions and notations. The Schrödinger operator, with all coefficients straightforwardly scaled out, reads as [2–6]

$$\hat{H} \stackrel{\text{def}}{=} -\frac{\mathrm{d}^2}{\mathrm{d}q^2} + q^N \qquad \text{for } N \text{ an even integer.} \tag{1}$$

(Obvious dependences upon the parameter N will be left implied.)

Over $L^2(\mathbb{R})$, $\hat{H}$ is a strictly positive operator; it has a purely discrete (nondegenerate) spectrum $\{\lambda_k\}_{k=0,1,2,\ldots}$, and $\lambda_k \uparrow +\infty$ according to the asymptotic Bohr–Sommerfeld law

$$b_0\lambda_k^\mu = 2\pi(k + \tfrac{1}{2}) + \mathrm{o}(1) \qquad k \to \infty \quad \text{in} \quad \mathbb{N} \tag{2}$$

which uses the classical action $b_0\lambda^\mu$ (at the energy λ, for the Hamiltonian $p^2 + q^N$), with

$$\mu \stackrel{\text{def}}{=} \frac{N+2}{2N} \quad \text{the } \textit{growth order} \qquad \text{and} \qquad b_0 = \frac{2\pi^{1/2}}{N}\Gamma\left(\frac{1}{N}\right)\Big/\Gamma\left(\frac{3}{2}+\frac{1}{N}\right). \tag{3}$$

Other dynamical constants describe the phases α, ω such that $\hat{H}_{|L^2(\omega\mathbb{R})}$ is unitarily equivalent to $\alpha\hat{H}_{|L^2(\mathbb{R})}$: this happens for $\omega = \alpha^{-1/2}$ and α in the cyclic group $\{\mathrm{e}^{\mathrm{i}\ell\varphi} \mid \ell = 0, 1, \ldots, L-1\}$, with

$$\varphi = \frac{4\pi}{N+2} \quad \text{the } \textit{symmetry angle} \qquad L = \frac{N}{2} + 1 \quad \text{the } \textit{symmetry order}. \tag{4}$$

† E-mail address: voros@spht.saclay.cea.fr

Since $\hat{H}$ commutes with the parity operator $\hat{P}$, it splits as $\hat{H}^{\pm} \stackrel{\text{def}}{=} \hat{H}(1 \pm \hat{P})$; the eigenfunctions of positive and negative parity carry the even and odd labels k respectively. *Spectral functions*, defined next, may also be split accordingly.

The spectral zeta functions ([4, 5]).

$$Z(s) \stackrel{\text{def}}{=} \sum_k \lambda_k^{-s} \qquad \left(\text{and} \quad Z^{\pm}(s) \stackrel{\text{def}}{=} \sum_{k\ {\text{even} \atop \text{odd}}} \lambda_k^{-s}\right) \qquad (\operatorname{Re} s > \mu) \tag{5}$$

extend *meromorphically* to all of $\mathbb{C}$, with only simple poles located at (or amongst) $s = (1-2j)\mu$, $j = 0, 1, 2, \ldots$. For $\operatorname{Re} s < \mu$ the series (5) diverge and the definitions must be regularized, according to equation (2); e.g., in each parity sector,

$$\sum_k{}^{*} \lambda_k^{-s} \stackrel{\text{def}}{=} \lim_{K\to+\infty} \left\{ \sum_{k<K} \lambda_k^{-s} - \frac{b_0\mu}{4\pi} \frac{\lambda_K^{-s+\mu}}{(-s+\mu)} + \frac{1}{2}\lambda_K^{-s} \right\} \qquad \text{if } \operatorname{Re} s > -\mu \tag{6}$$

with both k, K even for Z^+, odd for Z^-. In particular, $Z^{\pm}(0) = \pm\frac{1}{4}$ (using equation (2)), and

$$Z^{\pm\prime}(0) = \lim_{K\to+\infty} \left\{ -\sum_{k<K} \log\lambda_k + \frac{b_0}{4\pi}\lambda_K^{\mu}\left(\log\lambda_K - \frac{1}{\mu}\right) - \frac{1}{2}\log\lambda_K \right\} \qquad \text{for} \quad k, K\ {\text{even} \atop \text{odd}} \tag{7}$$

which in turn generates the (*zeta-regularized*) *determinants*, as $\det \hat{H}^{\pm} \stackrel{\text{def}}{=} \exp - Z^{\pm\prime}(0)$.

The spectral (or functional) determinants ([5, 7]). $D^{\pm}(\lambda) \stackrel{\text{def}}{=} \det(\hat{H}^{\pm} + \lambda)$: these are *entire* functions, more explicitly given by

$$D^{\pm}(\lambda) \equiv \exp[-Z^{\pm\prime}(0)] \prod_{k\ {\text{even} \atop \text{odd}}} (1 + \lambda/\lambda_k) \qquad (\text{for } \mu < 1, \text{ i.e., } N > 2) \tag{8}$$

$$= \exp\left[-Z^{\pm\prime}(0) - \sum_{n=1}^{\infty} \frac{Z^{\pm}(n)}{n}(-\lambda)^n \right]$$
$$(\text{for } \mu \neq 1, \text{ i.e., } N \neq 2, \text{ and } |\lambda| < \lambda_0). \tag{9}$$

The harmonic oscillator ($N = 2$) has a special status in the family: it is the solvable case but also the 'confluent' case, i.e., the growth order becomes *integer* ($\mu = 1$), and this invalidates several formulae in their generic form given here; in particular, $Z^{\pm}(1)$ are infinite and equations (8), (9) diverge, a valid substitute specification being

$$D^{\pm}(\lambda) = 2^{\pm 1/2}\sqrt{2\pi}\, 2^{-\lambda/2} / \, \Gamma\left(\frac{2 \mp 1 + \lambda}{4}\right) \qquad \text{for} \quad N = 2. \tag{10}$$

Hence $N = 2$, potentially the only elementary example for the formalism under review, is instead a pathological case.

The determinants admit semiclassical asymptotic expansions for $|\lambda| \to \infty$ ($|\arg\lambda| < \pi - \delta$). These have quite stringent forms, actually simpler for the full determinant $D = D^+D^-$ and the 'skew' ratio $D^{\mathrm{P}} \stackrel{\text{def}}{=} D^+/D^-$: [4]

$$\begin{aligned} \log D(\lambda) &\sim \sum_{j=0}^{\infty} a_j \lambda^{\mu(1-2j)} \qquad (N \neq 2) \qquad (\text{with } a_0 \equiv (2\sin\pi\mu)^{-1} b_0) \\ \log D^{\mathrm{P}}(\lambda) &\sim \tfrac{1}{2}\log\lambda + \sum_{r=1}^{\infty} d_r \lambda^{-(1+N/2)r}. \end{aligned} \tag{11}$$

The leading behaviours imply that D and $D^{\pm}$, as entire functions, are of order μ (the higher coefficients a_j, d_r are also computable term-by-term).

Exact functional relations ([8, 1]). The determinants satisfy a basic functional relation, obtained by an exact WKB calculation (and corresponding to the reflection formula for $\Gamma(z)$ when $N = 2$):

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\varphi/4} D^+(\lambda) D^-(\mathrm{e}^{\mathrm{i}\varphi}\lambda) - \mathrm{e}^{-\mathrm{i}\varphi/4} D^+(\mathrm{e}^{\mathrm{i}\varphi}\lambda) D^-(\lambda) \equiv 2\mathrm{i} \qquad (N \neq 2) \\ &\text{i.e.,} \quad \mathrm{e}^{\mathrm{i}\varphi/4} D_0^+ D_1^- - \mathrm{e}^{-\mathrm{i}\varphi/4} D_1^+ D_0^- \equiv 2\mathrm{i} \end{aligned} \tag{12}$$

upon introducing the generic shorthand notation $D_\ell(\cdot) \stackrel{\text{def}}{=} D(\mathrm{e}^{\mathrm{i}\ell\varphi}\cdot)$ (φ being the symmetry angle, and ℓ an integer mod L). Within identities like (12), all such subscripts are globally shiftable mod L, since λ can be freely rotated by $\mathrm{e}^{\mathrm{i}\varphi}$ throughout.

Equation (12) was found [1] as an equivalent form of a multiplicative coboundary formula linking the full and skew determinants, [5]

$$\mathrm{e}^{-\mathrm{i}\varphi/2} \frac{D^{\mathrm{P}}(\mathrm{e}^{\mathrm{i}\varphi}\lambda)}{D^{\mathrm{P}}(\lambda)} \equiv \exp[-2\mathrm{i}\Phi(\lambda)] \qquad \text{for} \quad \Phi(\lambda) \stackrel{\text{def}}{=} \arcsin[(D(\mathrm{e}^{\mathrm{i}\varphi}\lambda)D(\lambda))^{-1/2}] \tag{13}$$

$$\text{i.e.,} \quad \frac{D^{\mathrm{P}}_{\ell+1}}{D^{\mathrm{P}}_{\ell}} \equiv \exp \mathrm{i}(-2\Phi_\ell + \varphi/2) \qquad \ell = 0, 1, \ldots, L-1 \quad (\mathrm{mod}\ L) \qquad (N \neq 2). \tag{14}$$

This in turn entails a consistency condition upon D, embodied in this *cocycle relation* (of length L),

$$\sum_{\ell=0}^{L-1} \Phi_\ell(\lambda) \equiv L\varphi/4 \quad = \pi/2 \qquad \text{for even } N \qquad (N \neq 2). \tag{15}$$

Via the second equation (13), the full determinant D thereby inherits an autonomous functional equation, circularly symmetric of order L (and convertible to a polynomial form).

Spectral sum rules ([5]). Algebraic identities arise from expanding either functional relations (12) or (13) to all powers λ^n around $\lambda = 0$, with the help of equation (9); by equation (5), the output is an infinite sequence of sum rules connecting various spectral moments:

$$[Z'(0) =]\, Z^{+\prime}(0) + Z^{-\prime}(0) \equiv \log \sin \frac{\varphi}{4} \qquad (n = 0) \tag{16.0}$$

$$\sin \frac{\varphi}{4} Z^+(1) - \sin \frac{3\varphi}{4} Z^-(1) \equiv 0 \qquad (N \neq 2) \qquad (n = 1) \tag{16.1}$$

$$\sin \frac{3\varphi}{4} Z^+(2) - \sin \frac{5\varphi}{4} Z^-(2) \equiv \sin \frac{\varphi}{4} \left(2 \cos \frac{\varphi}{4} [Z^+(1) - Z^-(1)]\right)^2 \qquad (n = 2) \tag{16.2}$$

$$\vdots$$

$$\sin\left(n - \frac{1}{2}\right) \frac{\varphi}{2} Z^+(n) - \sin\left(n + \frac{1}{2}\right) \frac{\varphi}{2} Z^-(n) \equiv P_n\{Z^+(m), Z^-(m)\}_{1 \leqslant m < n} \qquad (\text{rank } n) \tag{16.n}$$

where P_n is a polynomial, homogeneous of degree n under the ruling that each $Z^\pm(m)$ is of degree m. The relative weights in the left-hand sides recur with period L in n, yielding $Z(n)$ whenever n is a multiple of L. For such $n = 0 \bmod L$, $Z(n)$ can be further reduced to a polynomial in the $\{Z(m)\}_{1 \leqslant m < n}$ only: this follows by directly expanding the closed functional equation for D itself, equation (15), in place of equation (12) (an option which, however, misses all the other identities of ranks $n \neq 0 \bmod L$). By contrast, no $Z^+(n)$ or $Z^-(n)$ ever comes alone on a left-hand side, which precludes similarly closed functional equations for D^+ or D^- separately.

Exact quantization conditions ([6, 1]). Exact quantization formulae à la Bohr–Sommerfeld for the spectrum ultimately derive from equation (12), as

$$\left.\begin{aligned} 2\Sigma_+(\lambda_k) &= k + \frac{1}{2} + \frac{\kappa}{2} \qquad k = 0, 2, 4, \ldots \\ 2\Sigma_-(\lambda_k) &= k + \frac{1}{2} - \frac{\kappa}{2} \qquad k = 1, 3, 5, \ldots \end{aligned}\right\} \qquad \text{with} \quad \kappa \stackrel{\text{def}}{=} \frac{N-2}{N+2} \tag{17}$$

where, excepting the case $N = 2$ (singular but solvable otherwise, to $\Sigma_\pm(\lambda) \equiv \lambda/4$),

$$\Sigma_\pm(\lambda) \stackrel{\text{def}}{=} \pi^{-1} \operatorname{Arg} D^\pm(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) \qquad (\lambda \geqslant 0) \tag{18$'$}$$

$$\operatorname{Arg} D^\pm(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) = \sum_{k' \,{\text{even} \atop \text{odd}}} \phi_{k'}(\lambda) \qquad \phi_{k'}(\lambda) \stackrel{\text{def}}{=} \arg(\lambda_{k'} - \mathrm{e}^{-\mathrm{i}\varphi}\lambda) \tag{18$''$}$$

i.e., $\operatorname{Arg} D^\pm$ denotes the determination of $\arg D^\pm$ which is continuous over the half-line $[0, \mathrm{e}^{-\mathrm{i}\varphi}\infty)$ and vanishing at $\lambda = 0$, and this is the sum of the angles $\phi_{k'}(\lambda) (\in [0, \pi))$ subtended by the vector $\overrightarrow{(0, \lambda)}$ at the points $\mathrm{e}^{\mathrm{i}\varphi}\lambda_{k'}$.

The system (17), (18) appears to specify the exact spectrum *self-consistently* within each parity sector. It amounts to a fixed-point condition for the (nonlinear) mapping defined by the application of equations (18) followed by (17), upon sequences of trial levels $\lambda_k^{(0)}$ that are *asymptotically correct* ([6, 1]: to be more precise, we require their compliance with equation (2)).

This fixed-point mechanism is moreover easy to implement numerically within some preset tolerance ε, which has to be nonzero to render the computational scheme finite-dimensional (by inducing a high-k cut-off): upon such truncations, then, the straightforward loop iterations of these maps appear to converge towards the correct spectra, consistently within $\mathrm{O}(\varepsilon)$, and at geometric rates [6]. Such a *constructive* enactment of equations (17), (18) works if, essentially, some form of *contractivity* holds for the underlying map. What we now have in this direction is no rigorous statement yet, but the numerical observation of such a behaviour in every case performed, plus partial analytical evidence: e.g., the mapping can be simplified by treating the spectrum as continuous [1], in which approximation its contraction factor is just κ—and $|\kappa| < 1$ by equation (17).

2. Extension to odd degrees and applications

The above framework will now be strengthened in two ways, both involving a violation of analyticity through the invocation of *odd* degrees N, in which case the confining potential becomes $|q|^N$.

From even to odd N. All previous results actually follow from solving a connection problem for the Schrödinger equation $(\hat{H} + \lambda)\psi(q) = 0$ along a *half-line*, e.g., from $q = 0$ to $+\infty$. The required *complex WKB* calculations were performed *exactly* [5] thanks mostly to the analyticity of the potential (whose homogeneity also eased the operations but at a more concrete level).

Consequently, these procedures apply to *odd* polynomials q^N as well, with the understanding that all results refer to the spectra specified by the same boundary conditions *on the half-line* $[0, +\infty)$ as in the case of even N: specifically, at $q = 0$, Neumann for positive parity, resp. Dirichlet for negative parity. But these in turn precisely select the positive, resp. negative parity spectra of the potential $|q|^N$ *also when N is odd*.

All previous formulae extend to odd N under this interpretation, with just one forced explicit change: the order L of the symmetry group, being an integer, cannot retain the value $N/2+1$ from equation (4), but becomes twice that instead:

$$L = N + 2 \qquad \text{for odd } N \tag{19}$$

while φ *stays unchanged*; consequently, the cocycle relation (15) becomes

$$\sum_{\ell=0}^{L-1} \Phi_\ell(\lambda) \equiv L\varphi/4 = \pi \qquad \text{for odd } N. \tag{20}$$

• Example: $N = 3$. We may test the extended formalism upon the confining homogeneous *cubic oscillator* (the operator $\hat{H} = -\mathrm{d}^2/\mathrm{d}q^2 + |q|^3$), for which

$$\mu = \tfrac{5}{6} \qquad \varphi = 4\pi/5 \qquad L = 5 \qquad b_0 = \frac{2^{2/3}\sqrt{3}}{5\pi}\,\Gamma\left(\frac{1}{3}\right)^3 = a_0 \qquad \kappa = \tfrac{1}{5}. \tag{21}$$

Among the several results which can be validated, we describe the exact quantization formulae (17), (18) as they probe the functional relation (12) the most sharply. So, we iterated equations (18), (17) (under a tolerance value $\varepsilon = 10^{-9}$) upon the input data consisting of each parity subsequence of the semiclassical spectrum $\lambda_k^{(0)}$ (given by $b_0\,\lambda_k^{(0)5/6} = 2\pi(k+\frac{1}{2})$), and we readily observed convergence to these lowest eigenvalues:

k	even	k	odd
0	1.022 9479	1	3.450 5627
2	6.370 2932	3	9.522 0764
4	12.870 297	5	16.369 373
6	20.000 879	7	23.745 471
8	27.592 421	9	31.530 790.

(22)

The convergence appears to be governed by contraction factors $\approx +0.233$ for positive parity (k, k' even) and $+0.189$ for negative parity (k, k' odd) (cf the continuous-spectrum prediction $= \kappa = \frac{1}{5}$).

Separately, we diagonalized the matrix of $\hat{H}$ in the orthonormal eigenfunction basis $\{\psi_n\}$ of the harmonic oscillator $-\mathrm{d}^2/\mathrm{d}q^2 + q^2$, using

$$\langle\psi_{n'}||q|^N|\psi_{n''}\rangle = \sqrt{\frac{2^{n'+n''}n'!n''!}{\pi}} \sum_{\substack{0\leqslant m'\leqslant n'/2\\ 0\leqslant m''\leqslant n''/2}} \left(-\frac{1}{4}\right)^{m'+m''} \frac{\Gamma(\frac{1}{2}(1+N+n'+n'') - (m'+m''))}{m'!m''!(n'-2m')!(n''-2m'')!} \tag{23}$$

for $n' = n''$ mod 2, zero otherwise; in each parity sector we applied several truncations (in the size range 20–40), and retained the eigenvalue figures as far as they were fully stable. These results then showed complete agreement with equation (22). (Here, the brute-force calculation might well be the less reliable one: equation (23) generates numerous large entries, through cancellations between even larger terms.)

The exact method is thus validated to an accuracy of 10^{-6}–10^{-7}, whereas its input (we used the semiclassical spectrum) was off by as much as 10^{-1} (for the ground state: $\lambda_0^{(0)} \approx 0.920\,791$).

• General N: an alternative exact approach exists for odd and even degrees alike: it takes the fully analytic potential q^N (nonconfining for N odd) and privileges a different spectral function, namely a *Stokes multiplier*—normalized here as $C(\lambda) \equiv \mathrm{e}^{-\mathrm{i}\varphi/4}c(\lambda)$ relative to [9, 2]. The present analysis connects to that approach and recovers its results, through the identity

$$C_0 \equiv (2\mathrm{i})^{-1}(\mathrm{e}^{\mathrm{i}\varphi/2}D_0^+D_2^- - \mathrm{e}^{-\mathrm{i}\varphi/2}D_2^+D_0^-) \qquad (\text{for } N \neq 2). \tag{24}$$

For instance, when $N = 3$, $C(\lambda)$ obeys a closed functional equation, $C_2C_3 - C_0 \equiv 1$ (the subscripts being shiftable mod 5; cf [9; 2 ch 5, section 27; 10]). Here, this equation is not just verified by the expression (24) (using equation (12)); it is thus even *solved* to the extent that the $D^{\pm}$ in equation (24) can be constructed by using the fixed-point equations (17), (18) to get their zeros (cf equation (22)) and then forming the Hadamard products (8). (By contrast, we do not know how to specify any of the other entire functions, like D or C, as directly).

(Note: remarkably, the same $N = 3$ functional-equation structure $C_2C_3 - C_0 \equiv 1$ also appears in integrable 2D field-theory models that involve dilogarithms [11], and in the first Painlevé function [12].)

Duality Within homogeneous problems, it is worthwhile asking which analytic potentials $q^N, q^{N'}$ might share the same rotation symmetry group, since the latter imprints the structure of spectral functions in a basic way. The answer is a *duality* relation,

$$\begin{aligned} &\varphi + \varphi' = 2\pi \\ &\text{implying} \quad \frac{1}{\mu} + \frac{1}{\mu'} = 2 \qquad L = L' \qquad \kappa' = -\kappa \\ &\text{and} \qquad NN' = 4. \end{aligned} \tag{25}$$

One solution is $N = N' = 2$: thus the harmonic oscillator is self-dual (but also singular). However, there exists one (and only one) other solution: the pair $N = 4$ (homogeneous quartic oscillator [5]) and $N' = 1$ (Airy equation [13]).

Hence, the Airy function (as the relevant solution of the linear potential) turns out to be conjugated with the spectral determinants of q^4! This form of duality may be weak (not implying exact links between the solutions of both problems) but still liable to reflect strong structural resemblances (now that the framework accommodates even and odd N on the same footing).

3. Airy function versus quartic oscillator

We will now basically scan properties of the Airy function ($N = 1$) [13] as they appear in the global setting of the $|q|^N$ problem, and especially analyse its duality with the quartic case ($N = 4$) [4, 5].

Already, (unlike $N = 2$,) both $N = 1$ and $N = 4$ share the status of *regular* values for the whole formalism as they give *noninteger* growth orders μ. Thus, the various constants corresponding to $N = 4$ are

$$\mu = \tfrac{3}{4} \qquad \varphi = 2\pi/3 \qquad L = 3 \qquad b_0 = \frac{\sqrt{2/\pi}}{3}\,\Gamma\left(\frac{1}{4}\right)^2 \qquad a_0 = b_0/\sqrt{2} \qquad \kappa = \tfrac{1}{3}. \tag{26}$$

Then (almost) all previous formulae also hold in their *raw (generic) form* for $N = 1$, upon simply resetting all the constants:

$$\mu = \tfrac{3}{2} \qquad \varphi = 4\pi/3 \qquad L = 3 \qquad b_0 = \tfrac{8}{3} \qquad a_0 = -\tfrac{4}{3} \qquad \kappa = -\tfrac{1}{3}. \tag{27}$$

Special features of the Airy case. When $N = 1$, the eigenvalues are those of the potential $|q|$, namely (up to sign) the *zeros* of the Airy function for negative parity and those of its derivative for positive parity (the connection with their usual notation [13] being $a_s \equiv -\lambda_{2s-1}$, $a'_s \equiv -\lambda_{2s-2}$, $s = 1, 2, \ldots$); the Bohr–Sommerfeld rule (2) (plus its corrections, omitted here) reproduces their known asymptotic behaviour. The corresponding spectral determinants are given by

$$D^-(\lambda) = 2\sqrt{\pi}\,\mathrm{Ai}(\lambda) \qquad D^+(\lambda) = -2\sqrt{\pi}\,\mathrm{Ai}'(\lambda). \tag{28}$$

(Normalization, the only nonobvious issue, is determined by adjusting the asymptotic forms (11), which are expressly without constant terms, against the known asymptotic expansions of Ai and Ai′ [13].)

Thus, the extension of the formalism to odd N reveals its *only* elementary regular example, given by the Airy functions (notwithstanding that their underlying potential $|q|$ is the most singular one at 0).

Still, one amendment is specially mandated when $N = 1$, namely a (standard) regularization [7] for those series and products which turn divergent as $\mu \geqslant 1$: thus, $Z^{\pm}(1)$ are now to be computed from equation (6) instead of (5), and equation (8) for $D^{\pm}$ accordingly gets replaced by

$$D^{\pm}(\lambda) = \mathrm{e}^{-Z^{\pm\prime}(0)+Z^{\pm}(1)\lambda} \prod_{k\ {\rm even \atop odd}} (1 + \lambda/\lambda_k)\mathrm{e}^{-\lambda/\lambda_k}. \tag{29}$$

The basic determinantal identities. We first compare the functional relations (12) and their consequences for the two conjugate cases,

$$\left.\begin{array}{lll} \mathrm{e}^{\mathrm{i}\pi/6}D^+(\lambda)D^-(\mathrm{j}\lambda) - \mathrm{e}^{-\mathrm{i}\pi/6}D^+(\mathrm{j}\lambda)D^-(\lambda) \equiv 2\mathrm{i} & \text{for} \quad N = 4 & (\varphi = 2\pi/3) \\ \mathrm{e}^{\mathrm{i}\pi/3}D^+(\lambda)D^-(\mathrm{j}^2\lambda) - \mathrm{e}^{-\mathrm{i}\pi/3}D^+(\mathrm{j}^2\lambda)D^-(\lambda) \equiv 2\mathrm{i} & \text{for} \quad N = 1 & (\varphi = 4\pi/3) \end{array}\right\}$$

$$(\mathrm{j} \stackrel{\mathrm{def}}{=} \mathrm{e}^{2\mathrm{i}\pi/3}). \tag{30}$$

The latter, by equation (28), is simply the classic Wronskian relation between $\mathrm{Ai}(\cdot)$ and $\mathrm{Ai}(\mathrm{j}^2\cdot)$, which both solve the same Airy equation:

$$\mathrm{W}[\mathrm{Ai}(\cdot), \mathrm{Ai}(\mathrm{j}^2\cdot)] \equiv (2\pi)^{-1}\mathrm{e}^{\mathrm{i}\pi/6}. \tag{31}$$

Here the Airy function $\mathrm{Ai}(z)$ acts in two ways, as solution *and* as spectral determinant of the Airy equation; this confusion of roles stems from the property (specific to $N = 1$) that the operator $\hat{H} + \lambda$ solely involves the combination variable $z = q + \lambda$.

For $N > 1$, by contrast, the determinants are not known to solve any second-order differential equation; while this remains to be confirmed in full generality, the fact is already certain for $N = 2$ (due to the presence of $\Gamma(z)$: cf equation (10), and [2, ch 5, section 27]). Still, the $N = 1$ and $N = 4$ functional relations are highly similar (duality!); but their different phase prefactors will suffice to create an essential distinction (as equation (33) will show).

If the discrete symmetry rotations are performed upon either of equations (30), a closed system of $L = 3$ equations follows:

$$\begin{array}{l} \mathrm{e}^{\mathrm{i}\varphi/4}D_1^+D_2^- - \mathrm{e}^{-\mathrm{i}\varphi/4}D_2^+D_1^- \equiv 2\mathrm{i} \\ \mathrm{e}^{\mathrm{i}\varphi/4}D_2^+D_0^- - \mathrm{e}^{-\mathrm{i}\varphi/4}D_0^+D_2^- \equiv 2\mathrm{i} \\ \mathrm{e}^{\mathrm{i}\varphi/4}D_0^+D_1^- - \mathrm{e}^{-\mathrm{i}\varphi/4}D_1^+D_0^- \equiv 2\mathrm{i}. \end{array} \tag{32}$$

With the D_ℓ^- as unknowns, this is a linear system, whose 3×3 determinant has the value

$$\Delta = 2\mathrm{i}\sin 3\varphi/4\, D_0^+D_1^+D_2^+. \tag{33}$$

• $N = 4$: $\varphi = 2\pi/3$, thus $\Delta \not\equiv 0$; then the odd spectral determinant $D^-(= D_0^-)$ can be solved as a functional of the even one (and vice versa), in rational terms:

$$D_0^- \equiv \frac{D_0^+ - \mathrm{j}^2D_1^+ - \mathrm{j}D_2^+}{D_1^+D_2^+} \qquad \text{and} \qquad D_0^+ \equiv \frac{D_0^- - \mathrm{j}D_1^- - \mathrm{j}^2D_2^-}{D_1^-D_2^-} \tag{34}$$

(the subscripts being shiftable mod 3).

• $N = 1$: $\varphi = 4\pi/3$, thus $\Delta \equiv 0$, a sharply different situation. Now the system cannot be solved for the D_ℓ^-, and the elimination process yields a (linear) relation among these instead, (and likewise for D^+),

$$\begin{aligned} &D_0^- + \mathrm{j}^2 D_1^- + \mathrm{j}^4 D_2^- \equiv 0 \qquad \text{i.e.,} \quad \mathrm{Ai}(\cdot) + \mathrm{j}\,\mathrm{Ai}(\mathrm{j}\cdot) + \mathrm{j}^2\,\mathrm{Ai}(\mathrm{j}^2\cdot) \equiv 0 \\ &D_0^+ + \mathrm{j} D_1^+ + \mathrm{j}^2 D_2^+ \equiv 0 \qquad \text{i.e.,} \quad [\mathrm{Ai}(\cdot) + \mathrm{j}\,\mathrm{Ai}(\mathrm{j}\cdot) + \mathrm{j}^2\,\mathrm{Ai}(\mathrm{j}^2\cdot)]' \equiv 0 \end{aligned} \tag{35}$$

hence both identities simply reflect the classic three-solution dependence relation for the Airy equation (but now in partnership with the nonlinear equations (34)).

The cocycle functional equations. Their comparison will already uncover a less obvious (new?) identity for the Airy functions.

• $N = 4$: in the quartic case, the cocycle relation (15) (of length 3) implies $\sin\Phi_2 = \cos(\Phi_0 + \Phi_1)$; expanding and squaring the $\cos\Phi_\ell$ away in succession yields $2\sin\Phi_0\sin\Phi_1\sin\Phi_2 + \sin^2\Phi_0 + \sin^2\Phi_1 + \sin^2\Phi_2 = 1$ and then, by the definition of Φ in equation (13), [5]

$$D_0 D_1 D_2 \equiv D_0 + D_1 + D_2 + 2. \tag{36}$$

• $N = 1$: the search for a nontrivial Airy counterpart of equation (36) must also be directed at the full spectral determinant, which now reads as

$$D = -4\pi\,\mathrm{Ai}\,\mathrm{Ai}' = -2\pi(\mathrm{Ai}^2)' \tag{37}$$

the relevant cocycle relation has again length $L = 3$, but this time it is equation (20); it now implies $\cos\Phi_2 = -\cos(\Phi_0 + \Phi_1)$ and then, proceeding just as before,

$$\begin{aligned} &D_0^2 + D_1^2 + D_2^2 - 2(D_1D_2 + D_2D_0 + D_0D_1) + 4 = 0 \\ &D_0 = -2\pi(\mathrm{Ai}^2)'(\cdot) \qquad D_1 = -2\pi(\mathrm{Ai}^2)'(\mathrm{j}^2\cdot) \qquad D_2 = -2\pi(\mathrm{Ai}^2)'(\mathrm{j}\cdot). \end{aligned} \tag{38}$$

Thus, $(\mathrm{Ai}^2)'$ exhibits a functional equation which is nonlinear, inhomogeneous, and with ternary symmetry like equation (36) (but rederivable by elementary means, *a posteriori*).

The Stokes multipliers. Here, their expression (24) plus the functional relations (12), (36) lead to diverging behaviours in the two cases.

• $N = 4$: the Stokes multiplier $C(\lambda)$ and the full determinant $D(\lambda)$ get related both ways,

$$D_0 \equiv C_1 C_0 - 1 \qquad \Longleftrightarrow \qquad C_0 \equiv (D_0 D_2 - 1)^{1/2} \tag{39}$$

(the subscripts being shiftable mod 3); hence C stands equivalent to D as a spectral function, and indeed it has a functional equation very close to equation (36), [10]

$$C_0 C_1 C_2 \equiv C_0 + C_1 + C_2. \tag{40}$$

• $N = 1$: $C(\lambda)$ degenerates to a constant ($\equiv 1$) [2] hence it has lost all spectral information. (More generally, $D(\lambda)$ might be unrelated to $C(\lambda)$ for odd N.)

Therefore, duality is devoid of content under this specific angle.

The spectral sum rules.

• $N = 4$: for the quartic case, the simple substitution $\varphi = 2\pi/3$ in equations (16) yields [5]:

$$Z'(0) = -\log 2 \tag{41.0}$$

$$\tfrac{1}{2}Z^+(1) - Z^-(1) = 0 \tag{41.1}$$

$$Z^+(2) - \tfrac{1}{2}Z^-(2) = \tfrac{3}{2}(Z^+(1) - Z^-(1))^2 \tag{41.2}$$

$$Z(3) = \tfrac{1}{6}Z(1)^3 - \tfrac{1}{2}Z(1)Z(2) \quad \text{etc.} \tag{41.3}$$

The left-hand side coefficients then recur with period $L = 3$ in n; all $Z(n)$ with $n = 3p$ reduce to polynomials in the $\{Z(m)\}_{1 \leqslant m < n}$ only, deducible directly from the functional equation (36) for D alone.

• $N = 1$: by setting $\varphi = 4\pi/3$ in equations (16), similar sum rules are produced for the *moments of the zeros* of $\mathrm{Ai}(-\lambda)$ for odd parity, resp. $\mathrm{Ai}'(-\lambda)$ for even parity (i.e., $Z^-(n) = \sum_s (-a_s)^{-n}$, $Z^+(n) = \sum_s (-a'_s)^{-n}$),

$$Z'(0) = -\log(2/\sqrt{3})\,[= Z^{+\prime}(0) + Z^{-\prime}(0)] \tag{42.0}$$

$$Z^+(1) = 0 \tag{42.1}$$

$$Z^-(2) = Z^-(1)^2 \tag{42.2}$$

$$Z(3) = Z^-(1)^3 - \tfrac{3}{2} Z^-(1) Z^+(2) = \tfrac{5}{2}\, Z(1)^3 - \tfrac{3}{2} Z(1) Z(2) \quad \text{etc.} \tag{42.3}$$

All properties of the quartic case persist ($L = 3$ again) but in addition here, the closure property of identities within zeta values of the same parity label holds not only for $Z(3p)$ but also for $Z^+(3p+1)$ and $Z^-(3p+2)$, thanks to the autonomous functional equations (35) for D^+ and D^- respectively.

Now, moreover, the Airy function is also a special function, whose Taylor series is accessible by other means [13], and this amounts to a full knowledge of the expansion (9). The 'spectral' sum rules above thereby get completed by the 'special' sum rules below, in which $\rho \stackrel{\text{def}}{=} -\mathrm{Ai}'(0)/\mathrm{Ai}(0) = 3^{5/6}(2\pi)^{-1}\,\Gamma(2/3)^2 \approx 0.729\,011\,133$:

$$Z^{+\prime}(0) - Z^{-\prime}(0) = -\log\rho \text{ (by definition)} \tag{43.0}$$

$$Z^-(1) = -\rho \tag{43.1}$$

$$Z^+(2) = 1/\rho \tag{43.2}$$

$$Z^-(3) = -\rho^3 + \tfrac{1}{2}\,(\Longleftrightarrow)\, Z^+(3) = 1 \quad \text{(etc).} \tag{43.3}$$

This last identity is exceptional for being so simple (the *inverse cubes of the zeros of* $\mathrm{Ai}'(-z)$ sum up to *unity*), and *rational*: all higher $Z^\pm(n)$ can be thus expressed when $N = 1$, but as *nontrivial* (rational) functions of ρ, hence are *never rational-valued* again.

Equations (42), (43) remind us of known sum rules for integer powers of zeros of the Bessel functions J_ν ([14], and references therein) but do not strictly correspond to them, since $\mathrm{Ai}(z)$ is a Bessel function of a *noninteger* power of z (and we know of no earlier explicit mention of any sum rule for Airy zeros). Both sets of identities, however, have the same abstract basis: they express the consistency between two full specifications of the given function, a Taylor series like equation (9) and a Hadamard product like equations (8), (29).

• For general $N \geqslant 2$, by contrast, the $N = 1$ explicit values have extensions *of the same form* only for $Z^{\pm\prime}(0)$ and $Z^\pm(1)$ ([4, 5 appendix C, D] stated the results for $N = 2M$ only [4], but this limitation is unwarranted):

$$Z^{-\prime}(0) = \log\left[(N+2)^{\frac{N\varphi}{8\pi}}\sqrt{\pi}\Big/\Gamma\left(\frac{\varphi}{4\pi}\right)\right] \qquad \left(\varphi = \frac{4\pi}{N+2}\right)$$

$$Z^+(1) - Z^-(1) = (2\sqrt{\pi})^{-1}\left[\frac{2}{N+2}\right]^{\frac{N\varphi}{2\pi}} \sin\frac{\varphi}{4}\,\Gamma\left(\frac{\varphi}{4\pi}\right)\Gamma\left(\frac{2\varphi}{4\pi}\right)\Gamma\left(\frac{3\varphi}{4\pi}\right)\Big/\Gamma\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right) \tag{44}$$

(the values for other parity labels follow through the identities (16.0), (16.1)).

The exact quantization of eigenvalues. The exact quantization mechanism (17), (18) is mainly governed by the symmetry angle φ, and it degenerates, with quantization becoming fully explicit, for $\varphi = \pi$ (harmonic case). With respect to this 'critical' system, the quartic oscillator

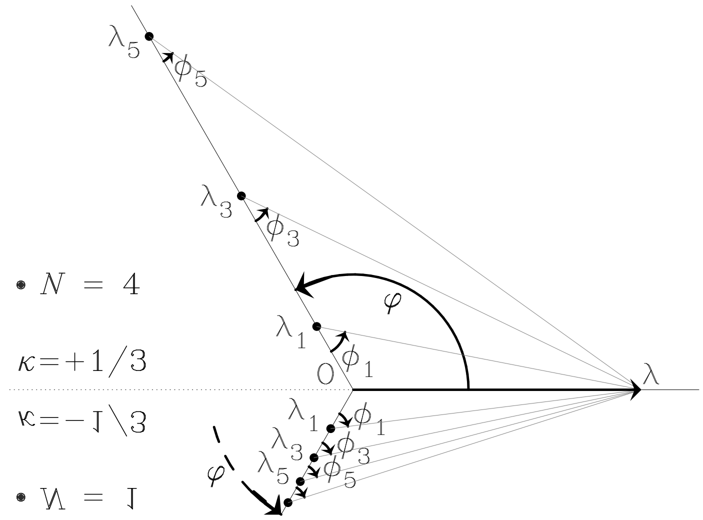

Figure 1. Geometrical depiction of the exact-quantization summands $\phi_{k'}(\lambda)$ (at $\lambda = 15.0$), in the odd spectral sector. Upper half-plane: for the quartic oscillator, using $\varphi = 2\pi/3$ in equation (18″); lower half-plane: for the zeros of $\mathrm{Ai}(-\lambda)$, using $\varphi = 4\pi/3$ in equation (45).

spectrum (with $\varphi = 2\pi/3, \kappa = +\frac{1}{3}$) and the Airy zeros (with $\varphi = 4\pi/3, \kappa = -\frac{1}{3}$) assume exactly mirror-symmetric positions; in particular, a key ingredient of the exact formula, the kernel function $\arg(\lambda' - \mathrm{e}^{-\mathrm{i}\varphi}\lambda)$ (associated with the linear 'flux operator' [6, 1]), takes opposite values in the two cases.

However, this symmetry is lost at some stage of the procedure (and we see no manifest link between the two resulting spectra either). Thus, the system (17), (18) is consistent and stable under iteration only when applied to spectra having the right growth order μ, which differs in the two cases ($\frac{3}{4}$ for $N = 4$, versus $\frac{3}{2}$ for $N = 1$). We therefore have the following consequences.

• $N = 4$: since $\mu < 1$, equations (17), (18) hold in their original form (figure 1, upper part). When applied to (definite-parity) input sequences $\lambda_k^{(0)}$ only subject to equation (2) (i.e., $b_0 \lambda_k^{(0)3/4} \sim 2\pi(k + \frac{1}{2})$, with b_0 from equation (26)), the iteration scheme exhibits contractive convergence to the exact eigenvalues, with ratios $\approx +0.392$ for positive parity (k, k' even) and +0.333 for negative parity (k, k' odd) [6].

• $N = 1$: now $\mu > 1$ hence the series in (18″) diverges like $(\lambda \sin\varphi)$ times the series (5) for $Z^{\pm}(1)$; its proper regularization, as dictated by equation (6) for $s = 1$ and equation (29), is then

$$\mathrm{Arg}\, D^{\pm}(\mathrm{e}^{-\mathrm{i}\varphi}\lambda) = \lim_{K \to +\infty} \left\{ \sum_{k' < K} \phi_{k'}(\lambda) - \lambda \sin\varphi \frac{2}{\pi} \lambda_K^{1/2} \right\} \qquad \text{for} \quad k', K \ {}^{\text{even}}_{\text{odd}}. \tag{45}$$

Here the summands $\phi_{k'}(\lambda)$, still defined as in equation (18″) but with $\varphi = 4\pi/3$, are all *negative and decreasing* (figure 1, lower part); it is now the counterterm $= +\lambda(\sqrt{3}/\pi)\lambda_K^{1/2}$ which outweighs the whole sum to produce a positive and increasing left-hand side, as required for equation (17).

Thereupon, the numerical scheme behaves for $N = 1$ as for the quartic case. When applied to each parity subsequence of the semiclassical spectrum $\lambda_k^{(0)}$ (given by $\frac{8}{3}\lambda_k^{(0)3/2} = 2\pi(k + \frac{1}{2})$), the iteration of equations (17), (18′), (45) exhibits convergence to the Airy zeros, with (roughly estimated) contraction factors −0.37 for positive parity (zeros of Ai′; k, k' even) and −0.25

for negative parity (zeros of Ai; k, k' odd). (However, whereas for $N > 2$ this scheme is comparatively quite efficient, for $N = 1$ it should not beat special-purpose algorithms for obtaining the Airy zeros; also in this case, the regularization (45) proceeds through large cancellations which strongly degrade the final accuracy.)

In conclusion, exact fixed-point quantization also works for the spectra of Airy zeros. Then, moreover, the conjugation symmetry shows this spectrum quantization to be *exactly* as nontrivial as that of the quartic oscillator, even though the Airy equation itself *is* more elementary than its quartic—or even its harmonic (and trivially quantized)—analogue.

Acknowledgments

We are very grateful to J Robbins (Bristol, GB) for his seminal suggestion of a possible link of the functional relation (12) with Wronskians (private discussion); to F W J Olver (Maryland, USA) for thorough bibliographical guidance on Airy functions; and to R Tateo (Saclay and Amsterdam) and Y Takei (RIMS, Kyoto) for acquainting us with [11] and [12], respectively.

References

[1] Voros A 1997 *Quasiclassical Methods (IMA Proc. Minneapolis, 1995)* vol 95 ed J Rauch and B Simon, IMA series (New York: Springer) pp 189–224

[2] Sibuya Y 1975 *Global Theory of a Second Order Linear Ordinary Differential Operator with a Polynomial Coefficient* (Amsterdam: North-Holland)

[3] Bender C M, Olaussen K and Wang P S 1977 *Phys. Rev.* D **16** 1740–8

[4] Voros A 1982 *The Riemann Problem, Complete Integrability and Arithmetic Applications (Lecture Notes in Mathematics vol 925*) ed D Chudnovsky and G Chudnovsky (Berlin: Springer) pp 184–208
Voros A 1980 Augmented version of *Nucl. Phys.* B **165** 209–36

[5] Voros A 1983 *Ann. Inst. Henri Poincaré* A **39** 211–338

[6] Voros A 1994 *J. Phys. A: Math. Gen.* **27** 4653–61

[7] Voros A 1987 *Commun. Math. Phys.* **110** 439–65

[8] Voros A 1992 *Zeta Functions in Geometry (Proc. Tokyo 1990) (Advanced Studies in Pure Mathematics vol 21)* ed N Kurokawa and T Sunada (Tokyo: Mathematical Society of Japan) pp 327–58

[9] Sibuya Y and Cameron R H 1973 *Symp. on Ordinary Differential Equations (Proc. Minneapolis) (Lecture Notes in Mathematics vol 312*) ed W A Harris Jr and Y Sibuya (Berlin: Springer) pp 194–202

[10] Sibuya Y 1984 On the functional equation $f(\lambda) + f(\omega\lambda) f(\omega^{-1}\lambda) = 1$, ($\omega^5 = 1$) *Proc. 38th Rencontre entre Physiciens Théoriciens et Mathématiciens* (Strasbourg, June 1984) vol 34, pp 91–103

[11] Zamolodchikov Al B 1991 *Phys. Lett.* B **253** 391–4
Gliozzi F and Tateo R 1996 *Int. J. Mod. Phys.* A **11** 4051–64 and references therein

[12] Kapaev A A 1989 *Diff. Eqns* **24** 1107–15

[13] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover) ch 10.4
Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic) ch 11 (Reprinted 1997 (Wellesley, MA: A K Peters))
Olver F W J 1998 *Airy Functions* ch 11, Digital Library of Mathematical Functions Project, National Institute of Standards and Technology, Gaithersburg, MD 20899-8910, USA (http://math.nist.gov/DigitalMathLib/)
Vallée O and Soares M 1998 *Les fonctions d'Airy pour la physique* (Paris: Diderot Éditeur)

[14] Watson G N 1952 *Theory of Bessel Functions* (Cambridge: Cambridge University Press) ch 15.41
Erdélyi A (ed) 1953 *Higher Transcendental Functions* vol II (New York: McGraw-Hill) ch 7.9